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New Ideas of Chaos in Physics

***Chaotic behaviour** in **physical systems** is described and examples given. Some implications for the limits of scientific prediction in areas where this applies are discussed—especially in **weather prediction**. Its bearing on a **reductionist approach** to such areas, its possibly constructive uses and its relevance to the debates about **physical determinism** are explored briefly.*

1. Newtonian Dynamics

In 1686, the foundation of Newtonian mechanics was laid when Newton published his *Principia*. His three laws of motion and his law of gravitation accounted for the motions of the planets around the sun as observed by Kepler and summarized by Kepler's three empirical laws decribing planetary orbits. The inverse square law of gravitation was called universal because it also accounted for the orbits of the satellites of Jupiter recently discovered by Galileo and for gravitational phenomena at the surface of the earth, such as tides.

In these first applications of Newton's laws just two bodies were involved, namely, the Sun and a planet, or a planet and a moon. A consequence of the law of gravitation is that for a body with spherical symmetry the effect of gravitational attraction by all the particles making up the body is equivalent to attraction by the whole mass concentrated at the centre. The equation for two such bodies can be solved analytically; exact solutions to any degree of accuracy can easily be realized.

Newton also recognized that the dynamical effects of the earth's rotation about its axis coupled with the effects of gravitation accounted for the shape of the earth with its flattened poles. He also recognized that this departure from spherical symmetry caused a perturbation in the gravitational force acting on the moon which partly explained the tendency of the moon's orbit to precess.

To solve for the motion in more complicated cases, analysts of planetary motions and of similar dynamical systems since Newton's time have developed perturbation theories in much greater detail. The method of perturbation theory is to begin from a simple first approximation such as an orbit satisfying Kepler's laws, to add successively perturbing effects to higher and higher degrees of approximation, finally seeking to prove rigorously that the procedure converges to a well determined limit. For instance, studies of the three-body problem which cannot be solved exactly have been concerned with the way in which the orbit of one body around another is perturbed by the presence of a third body. Famous names in the development of Newtonian mechanics during the eighteenth and nineteenth centuries were Euler, Lagrange, Laplace, Hamilton and Poincaré.

Such perturbation methods worked well in a wide range of cases in which convergence could be demonstrated. But there were some cases where there existed possibilities of approximate resonance between multiples of the different orbital periods of two interacting oscillations of the system and for which the perturbation method did not converge. Poincaré in 1903 recognized that the actual solutions in such cases might be highly dependent on the initial conditions. He wrote as follows: 'A very small cause which escapes our notice determines a considerable effect that we cannot fail to see, and then we say that the effect is due to chance. If we knew exactly the laws of nature and the situation of the universe at the initial moment, we could predict exactly the situation of that same universe at a succeeding moment. But even if it were the case that the natural laws had no longer any secret for us, we could still only know the initial situation approximately. If that enabled us to predict the succeeding situation with the same approximation, that is all we require, and we should say that the phenomenon had been predicted, that it is governed by laws. But it is not always so; it may happen that small differences in the initial conditions produce very great ones in the final phenomena. A small error in the former will produce an enormous error in the latter. Prediction becomes impossible, and we have the fortuitious phenomenon.'

2. Chaotic Behaviour

The unpredictable situations first recognized by Poincaré, in which given infinitesimally different starting points, systems can realize very different outcomes, have become known as chaotic behaviour. The rise in interest in such behaviour since the 1960s has resulted from the discovery that chaotic behaviour is widespread in nature and that it is commonplace even in simple systems which follow Newton's laws and which are not subject to disturbing influences.

Lighthill (1986) suggests three reasons why this discovery occurred in the 1960s. The first was the work which had continued from Poincaré's work on non-linear perturbation theory which demonstrated for very restricted ranges of initial conditions examples of chaotic motion in simple isolated systems. The second was the availability of electronic computers which made it much easier to compute solutions not just for small perturbations but also for much larger ones. Such work exposed many more examples of chaotic phenomena. For instance, it made it possible to compute the positions of billiard balls on a perfect table making perfect collisions and the sensitivity of those positions to the initial conditions. The sensitivity is surprising. As Crutchfield et al. (1986) point out, if the position of the cue ball after one minute is to be predicted accurately, effects in the initial stroke as small as the gravitational attraction of an electron at the edge of the galaxy have to be allowed for!

The third reason for the recent development of interest in chaotic behaviour has been because of research into the properties of turbulence associated with the motion of fluids. Osborne Reynolds, a century ago, showed that the regular flow of fluid through a pipe suddenly becomes turbulent (or chaotic) at a particular point when the force producing the motion becomes sufficiently large relative to the damping forces due to viscosity. He showed also that this turbulence has nothing to do with the random molecular movements on submacroscopic scales, but is a chaotic pattern of fluid movement on a macroscopic scale. Compared with the simple dynamical systems I have mentioned above such as a planet orbiting about the sun or even a set of balls on a billiard table which possess rather few degrees of freedom, a fluid possesses an almost unlimited number of degrees of freedom. Was the existence of this very large number of degrees of freedom a prerequisite for there to be turbulent motion? It was that sort of question which was addressed by a meteorologist, E. N. Lorenz in 1963 (see e.g. F. C. Moon, 1987). He wrote down a set of three ordinary differential equations to model thermal convection. They were the simplest set he could set up to describe the problem, and they possessed just three degrees of freedom. He discovered that all non-periodic solutions of these equations were bounded but unstable, i.e. they underwent irregular fluctuations without any element of randomness introduced from the outside. The term 'strange attractor' was introduced to describe this type of chaotic motion.

Since the 1960s many systems showing 'chaos' have been recognized and studied and there is now a large literature on the subject. In this short paper I shall first address one of the simplest examples, forced motion of a spherical pendulum, and then briefly look at a much more complex system, that of the weather and the problem of weather forecasting.

3. The Spherical Pendulum

A simple pendulum consists of a small bob attached to the end of a string of length . . . so that when the top end is fixed the movement of the bob is confined to a spherical surface of radius l. For small oscillations the natural period T_0 of sinusoidal oscillation is given by the well known expression $T_0 = 2\pi(l/g)^{1/2}$. If a periodic linear forcing motion is introduced at the point of suspension, the behaviour of the pendulum can be studied for various periods of the forcing oscillation. Several different types of behaviour can be distinguished.

First, if the driving period is far removed from the natural period of oscillation, the pendulum will swing in step parallel to the drive with small amplitude in a perfectly well determined manner.

Secondly, if the driving period is just below T_0 (around 0.989 T_0), when the drive is switched on the bob at first swings parallel to the drive. Because it is near resonance the amplitude increases and the motion develops a component perpendicular to the drive. After a complicated but repeatable series of fluctuations the motion settles down to a regular pattern, the bob moves in a nearly circular path, once around for each period of the drive.

Even this regular motion is not entirely predictable; whether the bob will circulate clockwise or anticlockwise cannot be predicted beforehand. If the experiment is performed many times the two senses will occur randomly. This lack of predictability arises because the component of motion perpendicular to the drive results from instability of the motion parallel to it. A solution to the governing equations exists with no perpendicular motion being present; this solution does not occur in practice because of the instability of this solution to small departures from perfectly parallel motion. Any small initial departure rapidly leads to much larger departures.

As the driving period increases towards and beyond T_0 more complicated but still regular patterns of motion occur until values of around 1.00234 T_0 are reached when the motion becomes 'chaotic'. Tritton (1986) and Lighthill (1986) describe these. Fig. 1 illustrates the range of variation of pattern of motions of the bob in the horizontal (x,y) plane. Notice that the motion is confined to a particular region of the diagram, a region which can be described as the 'attractor' for the system under these conditions. The diagram is compounded of many kinds of movements, for instance, those forming curves largely below the x-axis after an undetermined number of which a transition is made to movements largely above the x-axis again of an undetermined number before the mode of motion changes again. These motions vary not only randomly but discontinuously as a function of the initial conditions.

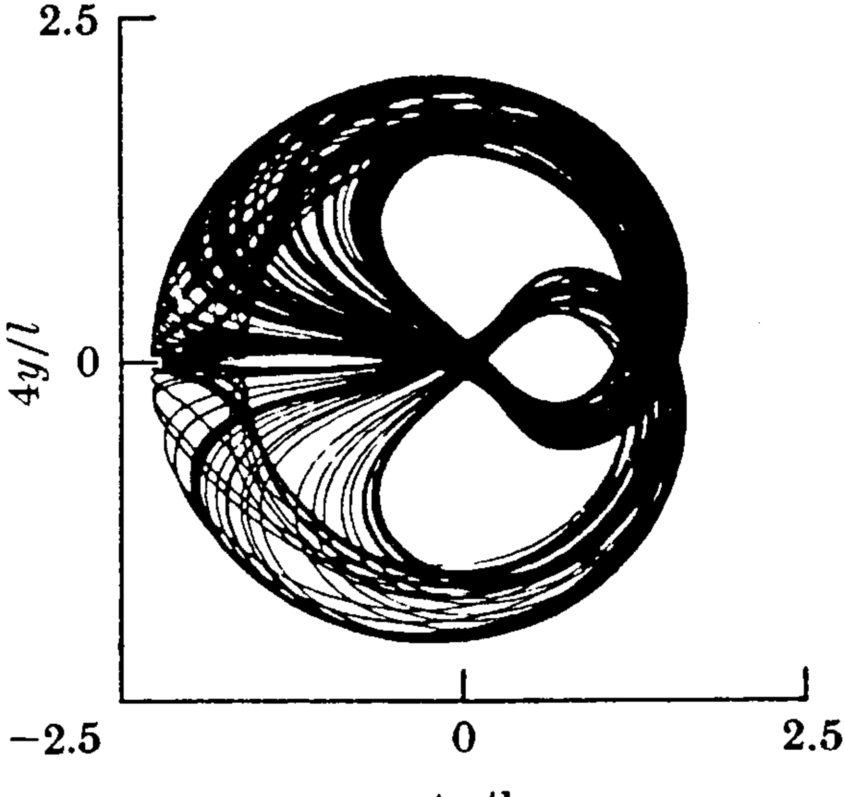

Figure 1

Suppose at some stage of the motion very precise details were available of the motion of the bob and of the forcing motion. Could the bob's subsequent motion be predicted? To begin with there would be good correspondence between predictability and observation. But after a time the predictability and the observation will diverge, the time before substantial divergence occurs being called the predictability horizon. If the initial conditions are more accurately defined, the predictability horizon will move further away, but not much so. Roughly speaking, as the number of decimal places increases in the definition of the initial conditions the predictability horizon changes approximately linearly. If, for instance, 4 decimal places (i.e. 0.01%) accuracy in the initial conditions enabled prediction for 1 minute, 12 decimal places or 1 part in a million million would provide for 3 minute prediction. This is because a property of chaotic systems is that neighbouring solutions diverge exponentially from one another.

A further point needs to be made about the type of system which exhibits chaotic behaviour: that is, that the governing equations must be non-linear. To obtain the expression for the period of a simple pendulum we assume that the amplitude is small such that $\sin \theta$ is approximated by θ. As the amplitude becomes larger this approximation no longer applies, the resonant period then depends on the amplitude of the swing. It is this feature that provides the non-linearity in the appropriate equations for the pendulum. Although physicists are very used to linear equations and indeed look for linear equations to provide approximate descriptions of any phenomenon or problem with which they are faced, nearly all systems in nature are non-linear and therefore fulfil one of the conditions for the exhibition of chaotic behaviour.

4. Weather Forecasting

I am including a section on applications to weather forecasting because they are my personal interest, and because they provide a good illustration of the subject of chaos.

I have already mentioned the work of E. N. Lorenz, who in addressing the problem of predictability in meteorology showed that the chaotic behaviour exhibited in fluid motion did not necessarily arise from the extremely large number of degrees of freedom associated with that motion, but was inherent in the simplest set of governing equations he could set up.

During the last twenty years or so the increase in accuracy of weather forecasts has largely occurred through the use of numerical models to carry out the basic prediction of the evolution of the atmosphere's motion on medium and large scales (i.e. scales greater than $\sim$ 100 km in horizontal extent). These models include the equations of fluid motion appropriate to the scales being modelled and representations of the physical processes (e.g. radiation transfer, convection, etc.) and the transfer of heat, momentum and water-vapour across the lower boundary. As improvements are made, either in the mathematics of the model, in the description of physical processes or in the power of the computer used (which enables the resolution of the model to be increased) experience has been, and still is, that better forecasts result. Forecasts over the British Isles for 5 days ahead are now useful and as good as 3-day forecasts were about 10 years ago.

However, we inevitably ask the question as to how much further predictability can be expected to improve. How much, for instance, does the accuracy of a forecast depend on the accuracy of the initial state from which the model integrations are made? This initial state is defined from observations from a wide variety of sources—aircraft, balloons, ships, land stations, satellites etc. An indication of this dependence can be obtained by using the model itself and integrating from initial states which differ by prescribed amounts. It is found that analysis errors tend to grow exponentially at first, doubling in amplitude about every two days and thereafter growing more slowly, reaching an asymptotic value after about 14 days. If this is a true representation of atmospheric behaviour the suggestion is that the atmosphere contains components of motion which show chaotic behaviour mixed with components which are more predictable.

A similar view of atmospheric behaviour comes from simulations of thermally-driven flows in rotating systems in the laboratory. Hide (1958) and his co-workers have demonstrated the existence under some circumstances of stable eddy patterns and under other circumstances of vacillating or unstable (chaotic) patterns of eddies. Hide (1980) has suggested similarity with the pattern of circulation in the atmosphere of Jupiter where a particularly stable eddy, the Red Spot, has been in existence for at least 300 years but is surrounded by a great deal of variable and turbulent flow. It is also possible that the 'blocking' patterns as they are called in the earth's atmospheric circulation which are associated with long periods of anticyclonic weather possess similarities with the stable eddy regimes observed in the laboratory simulations.

Other elements of the predictability of the atmospheric system are dependent on external factors such as coupling with the ocean circulation, changes in the land surface, in solar radiation or in atmospheric composition. Supposing, however, that none of these factors change, meteorologists are as yet at an early stage in their understanding of the degree of predictability which can be associated with different components of the atmosphere's circulation. Developments in our understanding of the physics of chaos will undoubtedly help.

5. A Simple Exercise

An illustration of the problem of atmospheric predictability may be provided by investigating a simple non-linear system described by the single first-order quadratic difference equation (see e.g. May (1976), Feigenbaum (1978)):

$$Y_{n+1} = aY_n - Y^2_n$$

where a is a constant. If $O \leqslant a \leqslant 4$ and $O \leqslant Y_o \leqslant a$, a sequence is generated in which $O \leqslant Y_n \leqslant a$ for all n.

This is also an equation which can describe chaotic behaviour in the dynamics of insect or animal populations.

For $a = 3.75$ and $Y_o = 1.5$ calculate a table of Y_n for values of n up to 30. Suppose there is an error in Y_o (representing the initial data) so that $Y_o = 1.501$. Work out a new sequence of Y_n.

Now set Y_o back to 1.5 and suppose that a is in error (representing an error in the model) so that $a = 3.751$. Again work out a new sequence of Y_n.

Finally, to simulate the effect of errors in mathematical procedure, starting from $a = 3.75$ and $Y_o = 1.5$ compute a series of Y_n in which each value of Y_n is rounded off to four significant figures.

Comment on the difference between the sequences of Y_n. In particular, for each case, how many stages are required for the error to grow by a factor of 10, 100, 1000?

6. Summary and Theological Implications

It is now recognized that a system as simple as a classic pendulum, where motion is entirely determined by Newtonian dynamics, can under a certain range of circumstances assume chaotic behaviour in which the behaviour of the system depends randomly and discontinuously on the initial conditions. Such behaviour, being associated with non-linear features of a system, occurs widely in nature.

For a chaotic system, a predictability horizon may be defined beyond which for a given accuracy in the specification of the initial conditions the behaviour of the system cannot be predicted. The predictability horizon only moves away linearly as the number of decimal places in the initial specification increases. Because of practical limitations, and eventually of Brownian motions associated with individual molecular movements, initial conditions can never be specified absolutely precisely so that the predictability horizon represents a fundamental limit to our ability to predict.

Considered from the point of view of classical physics it needs to be explained that this description of chaotic systems does not in any way destroy Newtonian ideas of causality or determinism. There is no suggestion, in the description from classical physics we have been following, that events in chaotic systems are undetermined. What we have pointed out is that their behaviour is so sensitive to initial conditions that their determination may involve complete knowledge in detail of the whole universe. However, if now we move on to include also the quantum mechanical description, as soon as the required specification of the initial conditions involves details of the movement of individual electrons or atomic nuclei, the Heisenberg uncertainty principle becomes relevant. We then come up against an inability, not only in practice but in principle, to specify with perfect precision the state of the system at any given time.

Are there theological implications to these results? Some, who are looking for ways for God to manoeuvre within a universe determined by scientific laws, may find in chaotic behaviour room for this intervention without as it were it being noticed. Such a view, however, makes God far too small. If we are to think of God as the Greatest Conceivable Being, and to follow the tenor of Biblical thought, then it is He that is the Creator and Sustainer of the Universe. It is He that by His moment by moment activity maintains everything in being (a theme expounded by Donald MacKay (1978)). Our scientific laws are then reflections of the orderliness of His activity.

Our experience as scientists and our expectation as believers in God is that the natural world should demonstrate an extremely high degree of orderliness, consistency and stability. But as Christian believers we also have an expectation that events in the natural world fit into a pattern of significance from the perspective of faith—something we commonly call God's providence. We need to believe that God is both big enough and clever enough to ensure both scientific consistency and at the same time providential significance.

The Christian will therefore be looking for a double consistency—scientific and providential (see MacKay (*loc. sit.*) and Houghton (1988) chapter 10). In this search he need not be influenced by the view he may have of the degree to which events in the natural world seem determined or predictable from the scientific point of view. I would argue therefore that there are no fundamental theological implications which arise from recent findings regarding chaotic systems and the realization that predictability horizons exist for many real world systems.

Having made the fundamental point, however, there are points of our scientific or theological *perspective* which may be affected by the results regarding chaotic systems. I suggest the following:

(1) The realization that the scientist's ability to predict the detailed behaviour of the natural world is much more limited than previously thought.

(2) An additional challenge to the reductionist view that a system can be understood by breaking it down into its component parts. Chaos demonstrates that a system can possess complicated behaviour that emerges as a consequence of the simple non-linear interactions of only a few components. There is no sense therefore in which it can be said that physics is complete when there is a detailed understanding of fundamental forces and particles. The interaction of components on one scale can lead to complex behaviour on a larger scale that in general cannot be deduced from knowledge of the individual components.

(3) In biological systems, nature may employ chaos constructively (Crutchfield et al (1987)). Through amplification of small fluctuations it can provide natural systems with access to novelty.

(4) Crutchfield et al. (1987) also speculate that ideas from 'chaos' may assist in thinking about processes in the brain. They close their article as follows:

> 'Even the process of intellectual progress relies on the injection of new ideas and on new ways of connecting old ideas. Innate creativity may have an underlying chaotic process that selectively amplifies small fluctuations and moulds them into macroscopic coherent mental states that are experienced as thoughts. In some cases the thoughts may be decisions, or what are perceived to be the exercise of will. In this light, chaos provides a mechanism that allows for free will within a world governed by deterministic laws.'

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